Indian Statistical Institute, Bangalore
B. Math II, First Semester, 2024-25
Back Paper Examination
Introduction to Statistical Inference

26.12.24 **Maximum Score 100** **Duration: 3 Hours**

1. (10) Let $X_1, \cdots, X_n$ be iid Poisson(λ) random variables. State with reason, which of the following is/are sufficient statistics for the parameter λ.
   (a) $\sum_{i=1}^n X_i$
   (b) $n - \sum_{i=1}^n X_i$
   (c) $\sum_{i=1}^n X_i - n\lambda$
   (d) X_1
   (e) $(X_1, \sum_{i=2}^n X_i)$

2. (25) Let ϕ_1 and ϕ_2 be bivariate normal densities with means zero, variances 1 and correlation coefficients 0 and 1/2 respectively. Suppose (X, Y) has the bivariate density $\frac{1}{2}(\phi_1 + \phi_2)$. Find the pdfs of X, Y and $X + Y$. Conclude that (X,Y) is not bivariate normal but the marginals are univariate normal.

3. (3+10+10+7) Let $X_i, \cdots, X_n$ be iid Geometric(p) random variables, that is, the pmf is

$$f(x) = (1-p)^{x-1}p \text{ for } x = 1, 2, 3, \cdots$$

   Let $T = \sum_{i=1}^n X_i$ and $\theta = p(1-p)$
   (a) Find an unbiased estimator of θ based on X_1.
   (b) Use Rao-Blackwell theorem to find an unbiased estimator $\hat{\theta}_U$ of θ based on T, that has lower variance than the estimator in part(a).
   (c) Find the MLE $\hat{\theta}_{MLE}$ of θ
   (d) Show that $\bar{X}$ is consistent for $1/p$. Hence conclude that both $\hat{\theta}_U$ and $\hat{\theta}_{MLE}$ are consistent for θ.

4. (12+3) Let $X_1, \cdots, X_n$ be iid from Exponential distribution with rate λ, that is, with pdf.

$$f(x) = \lambda \exp\{-\lambda x\}, \quad x > 0$$

   Let $\lambda_0 > 0$ be a fixed number.
   (a) Find the form of the generalized likelihood ratio test for testing $H_0 : \lambda = \lambda_0$ vs $H_1 : \lambda \neq \lambda_0$ at level α. Express the rejection region in terms of $\bar{X}, n, \alpha, \lambda_0$ and the Gamma distribution.
   (b) Can any test of this form be uniformly most powerful? Justify your answer.

5. (8+10+2) Let X be a single observation from the $\beta(\theta, 1)$ distribution.
   (a) Let $Y = -(\log X)^{-1}$. Evaluate the confidemce coefficient of the set $(y/2, y)$.
   (b) Find a pivotal quantity and use it to set up a confidence interval having the same confidence coefficient as the interval in part (a).
   (c) Compare the two confidence intervals.